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Abstract

Despite a plethora of studies in monetary economics regarding the study of inflation, interest rates, stock returns, and velocity of money, a model that helps to jointly characterize these interactions is still scarce in the literature. A key missing piece in most of the literature attempting such a characterization is idiosyncratic precautionary money demand, which is prevalent in the data. This paper presents a simple model where precautionary money demand arises due to heterogeneity in households' liquidity needs. In spite of its heterogeneous complexity, aggregation in the model is straightforward, this is one of the main contributions of the paper, and therefore an analysis of the models' implications can be undertaken when households' portfolio is composed of cash, government bonds, and equity. The empirical analysis is conducted separately for the time spans 1984.I-2007.IV and 2008.I-2019.IV. The model can capture important time-series properties that a model without the idiosyncratic feature is unable to achieve. However, the model falls short of providing an adequate match of some moments, especially in the second sub-sample of the analysis.

Keywords: Precautionary money demand, Portfolio allocation, Heterogeneity, Government bonds, Stock Market, Open market operations.

JEL Classification: E41,E51.

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1 Introduction

In general equilibrium monetary economics, there is an absence of model environments that consider an array of financial assets such as cash, government bonds, and equity for households while, incorporating precautionary money demand, which is prevalent in the data. Here I propose one, which is a modified version of the simplest Cash-in-Advance (CIA) model used in early studies of monetary economics.

I build a model where heterogeneity is key to obtaining precautionary money demand, as individuals face uninsurable idiosyncratic shocks to their marginal utility of consumption - liquidity shocks-, which realize after their portfolio decisions -between equity, government bonds, and cash- are made. Because of the CIA constraint in place, only cash serves at this stage. Hence, cash is held for precautionary motives that serve as a medium of exchange. The ex-post heterogeneity and the different wealth accumulation patterns through time, do not pose a computational difficulty because under standard Constant-Relative-Risk-Aversion (CRRA) preferences, the policy functions are linear in wealth and therefore aggregation is straightforward. These methods of solution are similar to those developed by Samuelson [1969] and Merton [1969], or more recently Angeletos [2007], Moll [2014] or Kiyotaki and Moore [2019], but were not exploited for the topic of interest. The application of this modeling feature to the CIA model with idiosyncratic precautionary money demand is one of the main contributions of the paper.

The main objectives of this study are to show how precautionary money demand can be introduced in a simple monetary model of the CIA type, and assess its empirical performance by contrasting its time-series properties against the data. Earlier important contributions to the CIA literature including Hodrick et al. [1991] and Giovannini and Labadie [1989] came short of providing a valid empirical validation of the CIA setup. A missing feature in this literature was precisely the precautionary motive for money demand. Telyukova [2013] documents how important this is in reality.

After the model is constructed, I calibrate it for the US at a quarterly frequency separately for the periods 1984.I-2007.IV and 2008.I-2019.IV. The method of calibration includes - for parameters that determine the dynamics of the model - a simulated method of moments. It is shown how the model delivers a rich array of predictions regarding the interaction between, inflation, nominal and real interest rates, stock market returns, and the velocity of money. When contrasted with the data, the model is favored in almost every dimension studied compared to the same model, but where the idiosyncratic component is shut down. Especially important is that the model can capture an important fraction of the variability of velocity and its correlation with other variables. I also find that the model's performance is, in general, better in the first sub-sample than in the second.

Related Literature

The model presented in this paper shares the heterogeneous feature of the class of heterogeneous agent models under uninsurable idiosyncratic shocks. Many studies in the so-called "Bewley tradition" (Bewley [1977]), for instance Hugget [1993], Aiyagari [1994], and Krusell and Smith [1998], explored different and important topics, but not precautionary money demand. Akyol [2004] does consider precautionary motives in such models, but he uses a different environment and studies only the stationary equilibrium and the optimality of the Friedman rule, another study along this line is Challe et al. [2017]. Although these papers consider shocks to income, there is a substantial strand of the literature, including Lucas [1980], Taub [1988], Taub [1994], Lucas [1992], and Atkenson and Lucas [1992], that consider shocks to the marginal utility of consumption, which this paper introduces. None of these contributions study the effects of introducing a precautionary motive in a monetary model. In addition, as mentioned before, simple aggregation is possible to obtain by applying insights from earlier literature, like Merton [1969] and Samuelson [1969].

This paper contrasts the model's time series properties against the data. Several studies did so, in many model environments. Boyle [1990] is an earlier contributor that investigates the relationship between asset prices and velocity of money in general equilibrium with money in the utility function, along these lines, Danthine and Donaldson [1986] study the relationship between inflation and asset prices, and a more recent contribution with this feature is Kraft and Weiss [2019]. Variable money velocity has also been incorporated by Svensson [1985] in a representative agent model, by adopting a timing convention where agents must decide on money holdings before the revelation of aggregate shocks. Another earlier contribution focusing on precautionary motives without incorporating idiosyncratic uncertainty is Giovannini [1989] who uses Svensson's setup and adds time-varying return distributions for the state variables. Precautionary motives arising due to information lags are also introduced by Lucas [1984]. As discussed by Hodrick et al. [1991], these modifications were unsuccessful, quantitatively. A dimension on which Svensson [1985] and Giovannini [1989] aimed to improve was the variability of velocity of money. While their environments allowed for variability in this measure, upon careful calibration and empirical evaluation, these models were quantitatively unsuccessful. Wang and Shi [2006] showed how a framework with costly search, by focusing on the extensive margin, can be quite successful along this dimension.

The closest previous works that incorporated precautionary money holdings in monetary models to the best of my knowledge are that of Wen [2015] and Telyukova and Visschers [2013]. Wen [2015] uses a related model to study the welfare cost of inflation. Telyukova and Visschers [2013] is similar because it puts at the forefront the issue of precautionary money demand and how it helps to account for the observed interaction of real and nominal variables in the business cycle, including the relevance of the velocity of money. This paper differs substantially in the type of model employed and the method of solution from both these important contributions. Furthermore, this paper introduces government bonds, open market operations, and the stock market which, is absent in their analysis.

The paper is organized as follows: Section 2 presents the model, Section 3 provides analysis and development of the theoretical model, Section 4 develops the calibration of the model, while Section 5 contrast the time series properties of the model with the data. Finally, Section 6 offer concluding remarks. Appendix A describes the data for the empirical part of the paper, while Appendix B present details of the modeling part including the Guess-and-Verify method of solution employed.

2 The Model

The economy is populated by a measure one, of individuals with CRRA utility, indexed by i, who maximize:[1]

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \theta_{i,t} \beta^t u(c_{i,t}), \quad u(c_{i,t}) = \frac{c_{i,t}^{1-\sigma}}{1-\sigma}. \tag{2.1}$$

$\theta_{i,t}$ is a preference shock. It is used to rationalize precautionary money demand related to liquidity needs, and as such it will be termed "liquidity shock" henceforth. I assume n possible cases with the following support:

$$\theta_{it} \in \{\theta_1, ..., \theta_j, ..., \theta_n\}, \tag{2.2}$$

[1] I want to thank professor Robert Lucas whose class notes on Monetary Theory were used with his permission as a basis for the development of the current model. Related seminal contributions are Lucas [1980], Lucas and Stokey [1983], and Lucas and Stokey [1987].

with $\theta_j < \theta_{j+1}$, and probability $\Pr(\theta_j) = \chi_j$, where $\sum_{j=1}^{n} \chi_j = 1$, so these shocks are i.i.d. over time and across individuals. $\mathbb{E}_0$ is the expectation operator that refers to the randomness induced by these idiosyncratic and aggregate shocks to be explained later.

At the beginning of period t individuals have total nominal wealth W_i. The asset market opens first, so agents decide on their portfolio. They choose to divide their wealth between equity shares (E_i), nominal government bonds (B_i), and money (M_i).[2] They would move then to a portfolio position satisfying:

$$W_i \geq Q_e E_i + Q_b B_i + M_i. \tag{2.3}$$

Q_e and Q_b are the dollar price of equity shares and one-period government bonds respectively. An equity share is a claim to the nominal dividend stream Py at the end of the period. Where P is the price level and y is the aggregate stochastic dividend to be specified below. They make this portfolio decision knowing all relevant state variables at that moment, except the liquidity shock θ_i.

After the portfolio decision is made, agents discover their shock θ_i. The household thereafter can be thought of as composed of two individuals, a producer, and a shopper, following Lucas' metaphor. The producer collects the endowment and sells it, obtaining $PE_i y$. The shopper takes the money held from the beginning of the period and goes to the market to purchase the

[2]The notational convention used in most of the paper is the following: lower case letters without subindex i will denote aggregate real variables and with subindex i individual variables. Upper case letters will denote nominal variables, again with subindex i for individual variables and without it for aggregates. Next period variables will be denoted with a prime.

consumption good, spending Pc_i, subject to the cash-in-advance (CIA) constraint:[3]

$$Pc_i \leq M_i. \tag{2.4}$$

At the beginning of the next period, equity shares can be sold at price Q'_e, obtaining $Q'_e E_i$. Agents will also carry to this period the cash from selling their endowment shares $PE_i y$, and redeem the government bonds purchased the last period B_i. There is also a possibility that they will have some unspent cash $M_i - Pc_i$. Therefore, nominal wealth at the beginning of next period is:

$$W'_i = Q'_e E_i + PE_i y + B_i + M_i - Pc_i. \tag{2.5}$$

The timing of the model can be viewed in Figure 1.

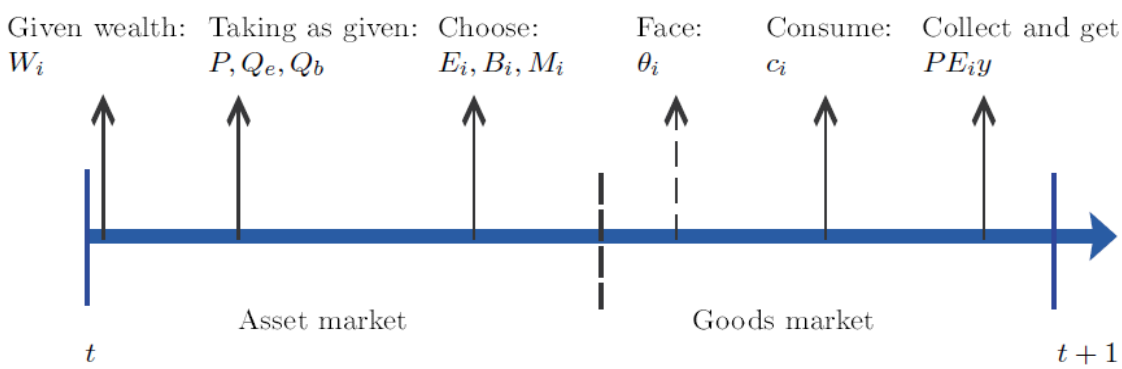

Figure 1: Timing. Not all information is revealed simultaneously. At the beginning of the period, households choose their portfolio E_i, B_i, M_i, without knowing the actual value of the liquidity shock θ_i. When it is realized, households can use money to consume c_i, subject to the CIA constraint. At this stage, they also collect their dividends $E_i y$ and sell them for cash at price P.

[3] I am not aware of a way to solve this model, by expanding the set of consumption goods to include cash and credit goods. As it will be clear in the next section, heterogeneity can be handled by imposing the strong assumption that all consumption needs cash. Then, when households substitute against the use of money, they will do so by substituting against consumption as well.

Government

Government is introduced by assuming that it pays for the bond issuing money or issuing new bonds:

$$B_s + M_s = M_s' + Q_b' B_s'. \tag{2.6}$$

Therefore, Open Market Operations (OMOs) are explicitly introduced.[4] And aggregate money supply follows:

$$M_s' = (1 + \gamma_m')M_s, \tag{2.7}$$

where γ_m' is the rate of growth of money supply. The dividend y is assumed to growth at rate γ_y:

$$y' = (1 + \gamma_y')y. \tag{2.8}$$

I assume that both γ_y and γ_m follow a joint stochastic process:

$$\gamma_y' = \bar{\gamma}_y + \rho_y(\gamma_y - \bar{\gamma}_y) + \rho_{ym}(\gamma_m - \bar{\gamma}_m) + \epsilon_y' \tag{2.9a}$$

$$\gamma_m' = \bar{\gamma}_m + \rho_m(\gamma_m - \bar{\gamma}_m) + \rho_{my}(\gamma_y - \bar{\gamma}_y) + \phi(x - \bar{x}) + \epsilon_m'. \tag{2.9b}$$

[4] This is in contrast to earlier literature in the CIA tradition where money is transferred directly to households. As discussed in Akyol [2004], in representative agent models, lump-sum transfers of money and retiring existing debt through open market operations are equivalent. However, in models with heterogeneous agents as this one, an open market operation would impact the economy differently as there is a nondegenerate distribution of agents with respect to bond holdings.

In addition, the innovations follow:

$$\begin{bmatrix} \epsilon'_y \\ \epsilon'_m \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \sigma_{ym} \\ \sigma_{ym} & \sigma_m^2 \end{bmatrix} \right). \tag{2.9c}$$

Hence, they are assumed to be of mean zero and variances σ_y^2 and σ_m^2 respectively, with covariance σ_{ym}.

Specification (2.9) is a vector autoregression for the growth rates of the dividend and money. $\bar{\gamma}_y$ and $\bar{\gamma}_m$ are their steady state values, respectively. Equation (2.9b) has nominal debt over GDP defined as $x \equiv B_s/(Py)$ in deviation from its long-run value $\bar{x}$. Because all money injections are conducted via OMOs, an equation assuming $\phi = 0$ will induce the model to display no equilibrium with standard methods such as Blanchard and Kahn [1980]. Assume, for instance an OMO, a negative ϵ'_m, by which government sells government bonds and withdraws cash. $\phi > 0$ ensures that if this OMO causes government bonds to surpass its long-run value next period, the rate of growth of money will tend to increase, at least partially reverting the previous OMO. ϕ measures how responsive is money growth to such deviations.[5] In (2.9), unless $\rho_{ym} = \sigma_{ym} = 0$, the economy is not a pure endowment one. In particular, monetary policy shocks may have real effects on the dividend, although the actual channel of transmission is not modeled.

Definition of Equilibrium

A competitive equilibrium is a sequence of price of equity, price of bonds, and price levels $\{Q_{e,t}, Q_{b,t}, P_t\}_{t=0}^{\infty}$, *and a sequence of aggregate stochastic dividend and money supply* $\{y_t, M_{s,t}\}_{t=0}^{\infty}$, *such that:*

[5]Therefore, the assumption $\phi > 0$ is important to ensure existence and uniqueness of the solution to the model. However, if ϕ is *too* large, an implosion may occur in government bonds and the model will display no solution. At a theoretical level, Lucas [1984] in section V on page 32, discusses the hardship of obtaining an equilibrium in a related monetary model with government bonds and OMOs.

1. *Taking as given prices, c_i, M_i, B_i, and E_i maximize individual's utility* (2.1) *subject to* (2.3), (2.4) *and* (2.5).

2. *Government satisfies its budget constraint, equation* (2.6).

3. *The financial and goods market clear:*

$$\int M_i di = M_s, \quad \int B_i di = B_s, \quad \int E_i di = 1, \quad \int c_i di = y. \tag{2.10}$$

3 Analysis

For details on the method of solution employed and related proofs, please see Appendix B.

The individual's problem

I assume and later verify that demand of the different assets are –across individuals– homogeneous fractions of their wealth:

$$M_i = z_m W_i, \quad Q_b B_i = z_b W_i, \quad Q_e E_i = (1 - z_b - z_m) W_i. \tag{3.1}$$

Where z_m and z_b are the shares of wealth in money and bonds, respectively. Using (3.1), and dividing by the price level, it is possible to write (2.5) and (2.4) as:

$$w'_i = R' w_i - R'_m c_i, \quad c_i \leq z_m w_i, \quad \text{where } R' = R'_e(1 - z_b - z_m) + R'_b z_b + R'_m z_m. \tag{3.2}$$

R' is the return on the portfolio, a weighted average of the (gross) returns on the three assets:

$$R'_e = \left(q'_e + \frac{y}{1+\pi'} \right) \frac{1}{q_e}, \quad R'_b = \frac{1}{(1+\pi')Q_b}, \quad R'_m = \frac{1}{1+\pi'}, \tag{3.3}$$

for equity, bonds, and money, respectively.[6] $q_e = Q_e/P$ is the real price of equity and π is the inflation rate. Returns defined in (3.3) are standard. R'_e includes the nominal dividend carried from the previous period, which in real terms is $y/(1+\pi')$. Also, because Q_b is the nominal price of bonds, its inverse defines the nominal interest rate: $1/Q_b = 1 + i$. Then $R'_b = (1+i)/(1+\pi')$ is the real rate on bonds.

Households must choose first the portfolio allocation not knowing yet their liquidity shock. Acting optimally, they consider the expected value of the shock in forming their optimal plans. The Bellman equation is:

$$\mathcal{V}(w_i) = \max_{z_m, z_b} \left\{ \sum_{j=1}^{n} \max_{c_i} \left[\theta_j u(c_i) + \beta \mathbb{E} \mathcal{V}'(w'_i)\right] \chi_j \right\}, \tag{3.4}$$

subject to (3.2).

The model is solved by first working out households decisions conditional on z_m, z_b and, on a given liquidity shock, that is by working out first the *inner* maximization in (3.4). The FOC reads:

$$\theta_j u'(c_i) \geq \beta \mathbb{E} \mathcal{V}'_w(w'_i) R'_m, \quad j = 1, ..., n. \tag{3.5}$$

With strict inequality if the CIA is binding. When the CIA is slack, there is a balance between the marginal utility of consumption and the discounted expected marginal value of wealth. In a first-order approximation, we can see the influence of the inflation rate. The higher the future inflation rate, the lower the expected return on cash, the more current consumption tends to rise. This occurs because consumption is paid with cash, and if the expected inflation increases, current consumption becomes cheaper. Most important, when θ_j is high, the desired

[6] In defining returns, we used the notational convention that, for example, R'_e is the *gross real* return on equity, the net return will be denoted $r'_e = R'_e - 1$. This notational convention differs from that defined in footnote 2.

consumption also increases, and it is more likely that CIA constraint binds. Idle balances, absent in original CIA models, will arise when (3.5) is satisfied with equality. For these individuals, their cash holdings chosen at the beginning of the period are ex-post large relative to the liquidity shock they get, so it does not pay to deplete their cash holdings and therefore reduce real wealth next period by a large amount.

To make progress in the solution of the model, a Guess-and-Verify strategy is employed for the value function:

$$\mathcal{V}(w_i) = \psi \frac{w_i^{1-\sigma}}{1-\sigma}. \tag{3.6}$$

Where ψ is an unknown stochastic variable whose process will be determined later.

We will assume from the outset that only individuals who receive the highest shock θ_n will satisfy their CIA constraint with equality, and for them (3.5) is satisfied with strict inequality.[7] Let $c_{i,j}$ be the policy function for agent i facing shock θ_j and $w'_{i,j}$ the associated next period wealth, then Appendix B shows that:

$$c_{i,j} = (1-\zeta_j) z_m w_i, \quad w'_{i,j} = R'_j w_i \quad j = 1, ..., n. \tag{3.7}$$

With $\zeta_j = 0$ for $j = n$ and:

$$R'_j = R'_e(1 - z_b - z_m) + R'_b z_b + R'_m \zeta_j z_m, \quad j = 1, ..., n, \tag{3.8}$$

is defined as the return on the portfolio after consumption. ζ_j on the other hand is a choice variable for the household, the savings rate out of cash. ζ_n is zero, meaning that CIA binds for

[7]It is natural to assume that only individuals that receive a relatively high liquidity shock, will bind their CIA constraint. That only those facing the highest shock are assumed to do so, is related to the calibration of this shock as shown in section 4. There, it is going to be shown that only 7% of individuals –that we will see correspond to the ones facing the highest shock– will deplete their cash holdings.

all agents facing the highest shock. All agents for which $\zeta_j > 0$, will have idle balances which will be subject to the inflation cost reflected in R'_m. $\zeta_j > 0$ is optimally chosen as to satisfy (3.5), which is –using the functional form for $u(\cdot)$ and the guess in (3.6)–:

$$\theta_j[(1-\zeta_j)z_m]^{-\sigma} = \beta \mathbb{E}\psi'(R'_j)^{-\sigma} R'_m, \quad j = 1, ..., n-1. \tag{3.9}$$

Policy functions in (3.7) are conditional on given θ_j. Households need to consider these to plan ex-ante when choosing z_m and z_b. The FOC with respect to z_m in (3.4) is:

$$\sum_{j=1}^{n} \theta_j u_c(c_{i,j}) \frac{\partial c_{i,j}}{\partial z_m} \chi_j = \beta \mathbb{E} \sum_{j=1}^{n} \mathcal{V}'_w(w'_i) \frac{\partial R'_j}{\partial z_m} \chi_j w_i, \tag{3.10}$$

where $u_c(\cdot)$ and $\mathcal{V}'_w(\cdot)$ are the derivatives of the respective functions and:

$$\frac{\partial R'_j}{\partial z_m} = R'_e - R'_m \left(\zeta_j + \frac{\partial \zeta_j}{\partial z_m} z_m\right), \quad j = 1, ..., n-1. \quad \frac{\partial R'_j}{\partial z_m} = R'_e, \quad j = n. \tag{3.11}$$

Equation (3.10) says that the expected marginal utility of cash –expected marginal utility services– is equal in the optimum to the expected discounted marginal cost. This cost is the induced change in lifetime value due to, choosing a more liquid portfolio in the margin, which foregoes the return on equity. If there are expected idle balances, however, these might obtain a return when carried for the next period, that is what the second term in the first equality in (3.11) is expressing. Note that in this computation, households consider how should ζ_j change with changes in z_m, which is obtained using the Implicit Function Theorem in (3.9), from which I obtain:

$$\frac{\partial \zeta_j}{\partial z_m} = \frac{\theta_j z_m^{-(\sigma+1)}(1-\zeta_j)^{-\sigma} + \beta \mathbb{E}\psi'(R'_j)^{-(\sigma+1)} R'_m (R'_e - \zeta_j R'_m)}{\theta_j (1-\zeta_j)^{-(\sigma+1)} z_m^{-\sigma} + \beta \mathbb{E}\psi'(R'_j)^{-(\sigma+1)} (R'_m)^2 z_m}. \tag{3.12}$$

The FOC with respect to z_b in (3.4) is:

$$\sum_{j=1}^{n} \theta_j u_c(c_{i,j}) \frac{\partial c_{i,j}}{\partial z_b} \chi_j = \beta \mathbb{E} \sum_{j=1}^{n} \mathcal{V}'_w(w'_i) \frac{\partial R'_j}{\partial z_b} \chi_j w_i, \tag{3.13}$$

where:

$$\frac{\partial R'_j}{\partial z_b} = R'_e - R'_b - R'_m \frac{\partial \zeta_j}{\partial z_b} z_m, \quad j = 1, ..., n-1. \quad \frac{\partial R'_j}{\partial z_b} = R'_e - R'_b, \quad j = n. \tag{3.14}$$

The LHS in (3.13) shows the expected marginal utility of bonds, which is still measuring liquidity services. In (3.10), consumption $c_{i,j}$ changes both because it is influenced by z_m directly and, because z_m influences ζ_j (for $j < n$). In (3.13) $c_{i,j}$ changes only indirectly through the change in ζ_j. So agents consider an expected change in idle balances that a portfolio with more bonds may enable. The discounted expected marginal cost now is influenced by the potential gain for carrying bonds to the future with return R'_b, and as before, with positive idle balances, these might carry a positive return when not used for consumption and carried to the next period.

The savings rate out of cash is influenced by z_b and it can be obtained again by the Implicit Function Theorem from (3.9) as:

$$\frac{\partial \zeta_j}{\partial z_b} = \frac{\beta \mathbb{E} \psi'(R'_j)^{-(\sigma+1)} R'_m (R'_e - R'_b)}{\theta_j (1-\zeta_j)^{-(\sigma+1)} z_m^{-\sigma} + \beta \mathbb{E} \psi'(R'_j)^{-(\sigma+1)} (R'_m)^2 z_m}. \tag{3.15}$$

Using the functional form for $u(\cdot)$ and the guess in (3.6), the FOC for z_m in (3.10) is:

$$\sum_{j=1}^{n-1} \theta_j [(1-\zeta_j) z_m]^{-\sigma} \left(1 - \zeta_j - \frac{\partial \zeta_j}{\partial z_m} z_m\right) \chi_j + \theta_n z_m^{-\sigma} \chi_n - \beta \mathbb{E} \psi' \sum_{i=1}^{n} (R'_j)^{-\sigma} \frac{\partial R'_j}{\partial z_m} \chi_j = 0. \tag{3.16}$$

Likewise, the FOC for z_b in (3.13) is:

$$\sum_{j=1}^{n-1} \theta_j [(1-\zeta_j) z_m]^{-\sigma} \left(-\frac{\partial \zeta_j}{\partial z_b} z_m \right) \chi_j - \beta \mathbb{E} \psi' \sum_{j=1}^{n} (R_j')^{-\sigma} \frac{\partial R_j'}{\partial z_b} \chi_j = 0. \tag{3.17}$$

The value of ψ still needs to be determined. The Appendix B shows that the Guess-and-Verify method gives:

$$\psi = \sum_{j=1}^{n} \theta_j [(1-\zeta_j) z_m]^{1-\sigma} \chi_j + \beta \mathbb{E} \psi' \sum_{i=1}^{n} (R_j')^{1-\sigma} \chi_j, \tag{3.18}$$

a recursion that determines ψ.

3.1 Aggregation

One of the virtues of the model developed is that all policy functions are linear in the relevant state and hence is straightforward to aggregate individuals in the economy. Aggregating the policy functions (3.7):

$$c \equiv \int c_{i,j} di = \sum_{j=1}^{n} (1-\zeta_j) \chi_j z_m w, \quad w' \equiv \int w_{i,j}' di = \sum_{j=1}^{n} R_j' \chi_j w. \tag{3.19}$$

Where w is aggregate real wealth. Regarding the demand of the assets of the economy, it is possible to go back to (3.1) taken in real terms to find aggregate demands:

$$\int m_i di \equiv m = z_m w, \quad \int b_i di \equiv b = \frac{z_b}{Q_b} w, \quad \int E_i di \equiv E = \frac{1 - z_b - z_m}{q_e} w, \tag{3.20}$$

which gives us the aggregate money demand, aggregate bonds demand, and aggregate equity demand, respectively.

3.2 Government

Recall that equation (2.6) describes the budget constraint of government. We can divide this equation by the price level and use the definition of inflation to get:

$$b_s + m_s = (1 + \pi') \left(m'_s + Q'_b b'_s \right), \tag{3.21}$$

where $b_s = B_s/P$ and $m_s = M_s/P$. Money creation in (2.7) can likewise be expressed in real terms as:

$$m'_s = \frac{1 + \mu'}{1 + \pi'} m_s. \tag{3.22}$$

The monetary growth rule (2.9b) can also be written with x expressed as $x = b_s/y$.

3.3 Velocity of money

In this model, a simple expression is available for velocity of money. By definition of velocity, we have that $\vartheta \equiv Py/M$. From the first equation in (3.19), and imposing goods market clearing:

$$\vartheta \equiv \sum_{j=1}^{n} (1 - \zeta_j) \chi_j. \tag{3.23}$$

Hence, how sensitive is velocity to fluctuations in dividends, interest rates, and so on, is determined on the different ζ_j. In the special case where with no idiosyncratic liquidity shocks (for instance, let all agents receive the same liquidity shock $\theta_j = \theta$, for all j) there is no precautionary money demand and $\zeta_j = 0$ for all j. In this case (3.23) gives $\vartheta = \sum_{j=1}^{n} \chi_j = 1$, a constant velocity. This clearly shows the importance of heterogeneity in the model to account for the variability of velocity that we observe in the data.

4 Calibration

4.1 Parameters pertaining the steady state

Important parameters are calibrated using the steady state of the model, which is explained in more detail in Appendix B.2. An important process to calibrate is the liquidity shock, agents are being buffeted continuously with this shock even in steady state.[8] I assume that there are $n = 5$ possible states for this shock. So there are ten parameters to calibrate: $\chi_1, ..., \chi_5$ and $\theta_1, ..., \theta_5$. The process of calibration is simplified by assuming that there is an underlying process for the liquidity shock with a continuous log-normal distribution:

$$\ln \theta_i \sim \mathcal{N}(0, \sigma_\theta^2). \tag{4.1}$$

Then we discretize this process using Tauchen's method for $n = 5$ states (Tauchen [1986]). Because the process for the liquidity shock was assumed –by necessity for the application of the law of large numbers and aggregation– i.i.d., the probabilities of the five states are $[0.07, 0.24, 0.38, 0.24, 0.07]$, independent of the value of σ_θ^2. Since we assumed that only the highest shock individuals are cash-constrained, this implies that 7% of agents deplete their cash holdings in each period. This accords well with the evidence of Telyukova and Visschers [2013].

[8]The steady state is non-stochastic at the aggregate level, but there is randomness at the individual level. Because a notion of non-stochastic steady state is considered for aggregates, in this situation the returns to all assets are the same. In this case, however, because of the assumption of a closed economy, there is no indeterminacy in portfolio allocations as these are given by the supply side. Money supply will grow at an exogenous rate and the government budget constraint will determine the supply of government bonds. Given households' wealth, the complement of their portfolio holdings will determine equity holdings. In open economy settings, the issue is more complicated, see for example Evans and Hnatkovska [2012].

Parameters that influence the steady state of the system are therefore only three: β, σ, and σ_θ^2.

In the non-stochastic steady state both R_e and R_b need to be the same, this rate labeled R satisfies:[9]

$$\bar{R} \equiv 1 + \bar{\gamma}_y + \frac{1}{(1+\bar{\pi})\bar{\hat{q}}_e} = \frac{1+\bar{i}}{1+\bar{\pi}}. \tag{4.2}$$

Equations determining optimal saving rates are:

$$\theta_j[(1-\bar{\zeta}_j)\bar{z}_m]^{-\sigma} = \beta\bar{\psi}\bar{R}_j^{-\sigma}\bar{R}_m, \quad j = 1, ..., n-1. \tag{4.3a}$$

Optimality of z_m:

$$\sum_{j=1}^{n-1}\theta_j[(1-\bar{\zeta}_j)\bar{z}_m]^{-\sigma}\left(1-\bar{\zeta}_j-\frac{\partial\bar{\zeta}_j}{\partial\bar{z}_m}\bar{z}_m\right)\chi_j + \theta_n\bar{z}_m^{-\sigma}\chi_n - \beta\bar{\psi}\sum_{i=1}^{n}\bar{R}_j^{-\sigma}\frac{\partial\bar{R}_j}{\partial\bar{z}_m}\chi_j = 0. \tag{4.3b}$$

The value for $\bar{\psi}$:

$$\bar{\psi} = \frac{\sum_{j=1}^{n}\theta_j[(1-\bar{\zeta}_j)\bar{z}_m]^{1-\sigma}\chi_j}{1-\beta\sum_{i=1}^{n}\bar{R}_j^{1-\sigma}\chi_j}. \tag{4.3c}$$

Government budget constraint:

$$\bar{z}_b(1+\bar{i}) + \bar{z}_m = (1+\bar{\pi})(\bar{z}_m+\bar{z}_b)(1+\bar{\gamma}_y). \tag{4.3d}$$

Goods market clearing equation:

$$(1-\bar{z}_b-\bar{z}_m) = \sum_{j=1}^{n}(1-\bar{\zeta}_j)\chi_j\bar{z}_m\bar{\hat{q}}_e. \tag{4.3e}$$

This is the strategy for calibration: First, $\bar{\gamma}_y$ and $\bar{\gamma}_m$ are taken from the data. In this case, equation (B.26) in Appendix B.2 also gives us the inflation rate $\bar{\pi}$. Second, I take an average

[9] As explained in Appendix B.1, variables with "hats" denote the variable normalized by the dividend y that grows continuously. Variables with a "bar" denote the steady state value.

of the returns on equity and bonds to get $\bar{R}$. Then, equations (4.2) deliver $\bar{i}$ and $\bar{q}_e$. Equations (4.3) then form a system of 8 equations for the unknowns. $\bar{\zeta}_1$, $\bar{\zeta}_2$, $\bar{\zeta}_3$, $\bar{\zeta}_4$, $\bar{z}_m$, $\bar{z}_b$, $\bar{\psi}$ and σ_θ^2. Finally, while β could have been set to an exogenous value, I found that the solution to the non-linear system of equations is easier to obtain if β is set to target how many times the value of consumption is held as precautionary money balances in the data, this measure is given by $\sum_{j=1}^{n} \bar{\zeta}_j \chi_j$ in the model. In the literature, there is a wide variation in this empirical measure. For example, Telyukova [2013] finds that the median household holds liquid assets in the order of 1.5 times the expenditure in consumption. However, for those households that borrow, this value falls to only 0.1.[10] A shortcoming of the data in her study for our purposes is that it excludes currency. Attanasio et al. [2002] for example, find for Italian data that currency holdings as a fraction of consumption is only in the order of 0.05. Hence, I set the target to a relatively low value of 0.25, to not artificially overemphasize the precautionary motive. Finally, I will perform the analysis for different values of σ.

The data used in this study and for calibration is described in Appendix A. I calibrate the model for a quarterly economy. In addition, I divide the whole analysis in the periods $1984.I - 2007.IV$ and $2008.I - 2019.IV$ throughout. This choice is made because of the effects of the Great Recession in both the asset markets and monetary policy in the latter period. As we will see, the time series properties of many variables appear to have changed indeed. For the first sub-sample, the values for the rate of growth of consumption and money are, $\bar{\gamma}_y = 0.005$ and $\bar{\gamma}_m = 0.013$ respectively, then inflation is $\bar{\pi} = 0.008$. Also, returns on bonds and equity are $\bar{R}_b = 0.002$ and $\bar{R}_e = 0.022$ respectively, then $\bar{R} = 0.012$. For the second sub-sample, I obtain $\bar{\gamma}_y = 0.002$ and $\bar{\gamma}_m = 0.015$ respectively, then inflation is $\bar{\pi} = 0.013$. For this sub-sample, returns on bonds and equity are $\bar{R}_b = -0.006$ and $\bar{R}_e = 0.017$ respectively, then $\bar{R} = 0.006$.[11] Table 1 shows the rest

[10] See Table 8 on page 1158. She uses the Survey of Consumer Finances and the Survey of Consumer Expenditures for the US for the period 2000-2002.

[11] In the empirical exercises below, I found that it matters little for average returns, whether I consider an equity premium by performing second or third-order approximations. Resolving the equity premium puzzle is

of the parameters and variables found by solving the non-linear system of equations (4.3) as explained before. As can be seen in the table, for high values of σ, the calibrated value of β is higher than one. This unusual odd value for "impatience" has been shown by Kocherlakota [1990] that is perfectly compatible with equilibrium in endowment economies. In the table (and the others that follow), we also present results for a model where the idiosyncratic shock is shut down. We assume that a single, representative agent faces only the highest liquidity shock, and therefore the CIA constraint always binds. These cases are presented under column RA in the tables. To distinguish from this case, in the tables, we show the results for the model developed with heterogeneity under the label HA, for heterogeneous agents.

4.2 Parameters involving dynamics

As for parameters involved in the dynamics of the model, those are given by the eight that parameterize the process (2.9):

$$\Gamma = [\rho_y, \rho_{ym}, \sigma_y, \rho_m, \rho_{my}, \sigma_m, \sigma_{ym}, \phi]. \tag{4.4}$$

In related previous studies, it was common to estimate the equivalent to process (2.9) directly using time series data for M2 and non-durable consumption expenditures for m and y respectively, see for example Hodrick et al. [1991] or Wang and Shi [2006] in a model with production. In our setup, however, because OMOs are considered explicitly, government bonds over GDP x influence money growth in equation (2.9b), and then the whole model economy is intertwined with the process (2.9). While in principle empirical measures for x could be proxied, it is unclear how to treat the potential bias that could arise by endogeneity issues or other biases that may

still an elusive matter. Early discussions on this issue in related models can be found in Backus et al. [1989] and Jerman [1998].

Table 1: Calibrated parameters and variables found in steady state

	$\sigma = 1$		$\sigma = 2$		$\sigma = 3$	
Parameter	HA	RA	HA	RA	HA	RA
A. 1984:I-2007:IV						
β	0.993	0.993	0.997	0.997	1.002	1.002
σ^2_θ	0.265	-	0.395	-	0.526	-
θ_1	0.589	-	0.454	-	0.349	-
θ_2	0.768	-	0.674	-	0.591	-
θ_3	1.000	-	1.000	-	1.000	-
θ_4	1.303	-	1.484	-	1.693	-
θ_5	1.697	1.697	2.203	2.203	2.865	2.865
$\bar{z}_m$	0.009	0.007	0.009	0.007	0.009	0.007
$\bar{z}_b$	0.016	0.013	0.016	0.013	0.016	0.013
$\bar{\hat{q}}_e$	135.460	135.460	135.460	135.460	135.460	135.460
$\bar{\hat{w}}$	138.989	138.199	138.989	138.199	138.989	138.199
$\bar{\psi}$	1.4E+02	2.3E+02	2.0E+04	4.2E+04	2.8E+06	3.6E-04
B. 2008:I-2019:IV						
β	0.997	0.997	0.999	0.999	1.001	1.001
σ^2_θ	0.252	-	0.383	-	0.514	-
θ_1	0.604	-	0.465	-	0.358	-
θ_2	0.777	-	0.682	-	0.598	-
θ_3	1.000	-	1.000	-	1.000	-
θ_4	1.287	-	1.466	-	1.672	-
θ_5	1.657	1.657	2.150	2.150	2.795	2.795
$\bar{z}_m$	0.004	0.003	0.004	0.003	0.004	0.003
$\bar{z}_b$	0.019	0.014	0.019	0.014	0.019	0.014
$\bar{\hat{q}}_e$	284.240	284.240	284.240	284.240	284.240	284.240
$\bar{\hat{w}}$	290.929	289.431	290.929	289.431	290.929	289.431
$\bar{\psi}$	3.0E+02	4.8E+02	8.7E+04	1.8E+05	2.0E-04	1.2E-04

Notes: Obtained steady state values by solving the system (4.3) for different σ. Under columns HA, it is shown results for the heterogeneous agent model developed. A similar system of equations for the representative agent case in which, all agents face θ_5 with certainty is shown under column RA. Panel A shows results for the period 1984.I-2007.IV while Panel B presents results for the period 2008.I-2019.IV.

occur by estimating only a part of the model economy. We chose then to use the model itself to calibrate Γ. Let the empirical moments to target be (here T denotes the transpose of the vector):

$$M = [corr(\gamma_y', \gamma_y), corr(\gamma_y', \gamma_m), sd(\gamma_y), corr(\gamma_m', \gamma_m), corr(\gamma_m', \gamma_y), sd(\gamma_m), corr(\gamma_y, \gamma_m)]^T, \quad (4.5)$$

which are directly related to variables in (2.9). Let $\mathcal{M}(\Gamma)$ be the equivalent model-simulated moments.[12] Then Γ is calibrated in such a way as to:

$$\min_{\Gamma} [M - \mathcal{M}(\Gamma)]^T [M - \mathcal{M}(\Gamma)]. \quad (4.6)$$

The calibration is again performed for the two sub-samples considered. Calibrated parameters for the process (2.9) are presented in Table 2. The table shows how these calibrated parameters change across sub-samples. These changes, of course, reflect variations in the actual targeted moments (4.5). Table 3 shows how these targeted moments differ across sub-samples. There are even changes in the sign of correlations, such as the autocorrelation of dividend growth $corr(\gamma_y', \gamma_y)$, and the contemporaneous correlation of growth of dividend with the growth of money $corr(\gamma_y, \gamma_m)$. The table also shows that for each sub-sample the procedure of calibration gives close model-simulated moments to their empirical counterparts.

[12] These moments depend on all parameters of the model, not only those on (4.4), but we make this explicit on Γ as these will be the parameters to be calibrated in this stage. The simulated moments are obtained by solving the linearized system of rational expectations displayed in equations (B.22) in Appendix B.1, using Blanchard and Kahn [1980] methods. I used Dynare for such purpose.

Table 2: Calibrated parameters with moment matching

	$\sigma = 1$		$\sigma = 2$		$\sigma = 3$	
Parameter	HA	RA	HA	RA	HA	RA
A. 1984:I-2007:IV						
ρ_y	-0.0275	-0.0275	-0.0275	-0.0275	-0.0275	-0.0275
ρ_{ym}	-0.0796	-0.0728	-0.0728	-0.0606	-0.0809	-0.0782
σ_y	0.0038	0.0034	0.0034	0.0034	0.0040	0.0036
ρ_m	0.3606	0.1873	0.3238	0.2097	0.3141	0.2243
ρ_{my}	-0.3761	-0.2936	-0.4152	-0.3705	-0.3758	-0.2703
σ_m	0.0042	0.0036	0.0040	0.0045	0.0042	0.0036
σ_{ym}	0.0715	0.0795	0.0724	0.0765	0.0738	0.0793
ϕ	0.1218	0.1053	0.0931	0.0870	0.0916	0.1033
B. 2008:I-2019:IV						
ρ_y	0.2326	0.2326	0.2326	0.2326	0.2326	0.2326
ρ_{ym}	-0.1986	-0.1952	-0.1949	-0.1996	-0.1999	-0.1999
σ_y	0.0029	0.0029	0.0029	0.0029	0.0029	0.0029
ρ_m	0.3740	0.0831	0.3124	0.2232	0.3176	0.2518
ρ_{my}	-0.2381	0.0345	-0.1480	-0.1287	-0.1549	-0.2201
σ_m	0.0057	0.0049	0.0059	0.0054	0.0059	0.0053
σ_{ym}	0.0123	0.0143	0.0120	0.0131	0.0119	0.0133
ϕ	0.0806	0.0447	0.0216	0.0337	0.0167	0.0353

Notes: Calibrated parameters according to (4.6) for different values of σ. Under columns HA, it is shown the results for the heterogeneous agent model developed. The same objective function and empirical targets were used for the representative agent version of the model, results for which are presented in columns labeled RA. Panel A shows calibrated parameters for the period 1984.I-2007.IV while Panel B presents results for the period 2008.I-2019.IV.

Table 3: Targeted moments and model's performance

Variable	Data	$\sigma = 1$		$\sigma = 2$		$\sigma = 3$	
		HA	RA	HA	RA	HA	RA
A. 1984:I-2007:IV							
$corr(\gamma'_y, \gamma_y)$	-0.031	-0.031	-0.031	-0.031	-0.031	-0.031	-0.031
$corr(\gamma'_y, \gamma_m)$	-0.105	-0.105	-0.105	-0.105	-0.105	-0.105	-0.105
$sd(\gamma_y)$	0.003	0.004	0.003	0.003	0.003	0.004	0.004
$corr(\gamma'_m, \gamma_m)$	0.284	0.284	0.284	0.284	0.284	0.284	0.284
$corr(\gamma'_m, \gamma_y)$	-0.268	-0.268	-0.268	-0.268	-0.268	-0.268	-0.268
$sd(\gamma_m)$	0.005	0.005	0.005	0.005	0.006	0.005	0.005
$corr(\gamma_y, \gamma_m)$	0.037	0.037	0.037	0.037	0.037	0.037	0.037
B. 2008:I-2019:IV							
$corr(\gamma'_y, \gamma_y)$	0.299	0.299	0.299	0.299	0.299	0.299	0.299
$corr(\gamma'_y, \gamma_m)$	-0.440	-0.440	-0.440	-0.440	-0.440	-0.440	-0.440
$sd(\gamma_y)$	0.003	0.003	0.003	0.003	0.003	0.003	0.003
$corr(\gamma'_m, \gamma_m)$	0.360	0.360	0.360	0.360	0.360	0.360	0.360
$corr(\gamma'_m, \gamma_y)$	-0.125	-0.125	-0.125	-0.125	-0.125	-0.125	-0.125
$sd(\gamma_m)$	0.007	0.007	0.007	0.007	0.007	0.007	0.007
$corr(\gamma_y, \gamma_m)$	-0.165	-0.165	-0.165	-0.165	-0.165	-0.165	-0.165

Notes: This table shows the empirical moments M (all data is logged and HP filtered) in the first column and the calibrated moments $\mathcal{M}(\Gamma)$, according to (4.6), in the rest of the columns for different values of σ. Under columns HA, it is shown the results for the heterogeneous agent model developed. The same objective function and empirical targets were used for the representative agent version of the model, results for which are presented in columns labeled RA. Panel A shows calibrated parameters for the period 1984.I-2007.IV while Panel B presents results for the period 2008.I-2019.IV.

5 Time series properties of the model

In this section, I make an empirical assessment of the model by examining its time series implications. Focusing on standard deviations, correlations, and autocorrelations. As mentioned before, the data used is described in Appendix A.

5.1 Volatility

I start with Table 4 that shows standard deviations of some variables. The volatility of velocity is an important variable to look at because typical models in the CIA tradition were disregarded based on its counterfactual lack of variability. In the data, which is logged and HP filtered throughout, we find that the standard deviation is between 0.014 and 0.016 depending on the sub-sample. As can be seen in the first row of both panels, the model can capture between 30 and 70 % of that variability. This comes in stark contrast with the RA model, where the volatility of velocity is null as the CIA always binds. The model is also quite successful to capture the volatility of the interest rate i, and the net (ex-post) real rate r_b. For these variables, even the RA model performs quite successfully. As for the volatility of the net return on equity r_e, the model fails by a large margin. Finally, the model can capture quite well the volatility of inflation, which is perhaps surprising due to the lack of price rigidities in the model. The RA model for this variable does not perform as well as the model with idiosyncratic liquidity shocks.

5.2 Correlations

Table 5 shows different correlations among variables of the model. In general, the performance of the model is satisfactory for the first sub-sample and less so for the second sub-sample. Let

Table 4: Standard deviations

		$\sigma = 1$		$\sigma = 2$		$\sigma = 3$	
Variable	Data	HA	RA	HA	RA	HA	RA
A. 1984:I-2007:IV							
$sd(\vartheta)$	0.014	0.007	0	0.005	0	0.004	0
$sd(i)$	0.003	0.001	0.003	0.001	0.004	0.002	0.003
$sd(r_b)$	0.004	0.004	0.005	0.004	0.006	0.005	0.005
$sd(r_e)$	0.052	0.004	0.004	0.004	0.004	0.005	0.004
$sd(\pi)$	0.003	0.004	0.006	0.004	0.007	0.005	0.006
B. 2008:I-2019:IV							
$sd(\vartheta)$	0.016	0.011	0	0.008	0	0.004	0
$sd(i)$	0.001	0.001	0.005	0.002	0.004	0.002	0.004
$sd(r_b)$	0.005	0.004	0.007	0.005	0.008	0.007	0.010
$sd(r_e)$	0.060	0.003	0.003	0.004	0.005	0.006	0.007
$sd(\pi)$	0.005	0.004	0.008	0.005	0.008	0.006	0.008

Notes: The table presents standard deviations for velocity, the nominal interest rate, the real rate ($r_b = (1+i)/(1+\pi) - 1$), and the return on equity ($r_e = R_e - 1$). The first column shows (logged) HP data statistics and the rest of the columns show theoretical moments for different values of σ. Under columns HA, it is shown the results for the heterogeneous agent model developed. The model statistics for the representative agent version of the model is presented in columns labeled RA. Panel A shows moments for the period 1984.I-2007.IV while Panel B presents moments for the period 2008.I-2019.IV.

me divide the discussion for both sub-samples.

Sub-sample 1984.I-2007.IV

The low correlation of velocity with consumption growth and money growth (−0.01 and −0.14, respectively) is captured by the model with low σ. The correlation of velocity with the nominal interest rate is quite high in the data (0.73) and the model predicts a perfect correlation. The model predicts this as portfolio allocations at the beginning of the period are frictionless, and agents react instantly to movement in the nominal rate. However, a high correlation among these variables, a desirable feature for a monetary model, is very difficult to obtain in older literature in the CIA tradition. The correlation of velocity with the real interest rate is 0.29 and lower in the model, but the low correlation of velocity with the real return on equity is close to the data. The correlation of velocity with inflation in the model is close to the data as well (0.24 in the data and between 0.15 and 0.20 in the model).

Correlation of inflation with consumption growth is overstated in the model (−0.32 and around −0.9 in the model), this is one of the few instances that the RA model performs somewhat better. The correlation of inflation with money growth is completely missed by the model and by the RA case. The correlation of inflation with the nominal interest rate is captured well by the model (in the data is 0.26 and the model gives values in the range of 0.15 and 0.20), while the RA case overstates it. The correlation of inflation with the real interest rate is also close to the data. The correlation of inflation with the return on equity is captured in the sign, but it is overstated in the model. This negative relationship was documented in the literature before, see for example Geromichalos et al. [2007].

Finally, we examine the correlation of the return on equity with several variables.[13] While

[13] Among previous papers that examined the relationship between the growth of money and asset prices is the work of Carmichael [1989]. Later literature that considered the stock market in their model environments such

the sign of the correlation of return on equity with consumption growth is correct, the model overstates it. This correlation is 0.21 in the data and close to unity in the model, also in the RA case. The correlation of return on equity with money growth is well captured for low levels of risk aversion, for the case $\sigma = 1$ the model gives 0.04, almost matching the 0.06 in the data. The correlation of return on equity with the real interest rate is overstated, but the sign is correct. The data gives a low correlation of 0.15, while the model gives correlations close to one. The RA model again performs somewhat better for this case, giving correlations around 0.40. Finally, the correlation of return on equity with the nominal interest rate is effectively captured by the model. The data for this correlation is -0.04 and the model for $\sigma = 1$ gives -0.03. Here, the RA gives values very different to the data.

Sub-sample 2008.I-2019.IV

For this sub-sample, the performance of the model is much less satisfactory, in general. Although still, the correlation of velocity with some variables is already an improvement over the RA model. We can notice that the correlation of velocity with consumption growth is captured well for relatively high values of risk aversion (in the data the correlation is 0.30 and the model delivers 0.33 for $\sigma = 3$). The correlation of velocity with money growth is closer to the data again for high values of risk aversion. The correlation of velocity with the nominal interest rate is lower in the data for this sub-sample (0.59 versus 0.73 for the previous sub-sample), but still quite high, and the model predicts a perfect correlation. The correlation of velocity with both the real interest rate and return on equity are not matched now by the model. Data gives correlations of -0.27 and -0.12 respectively while the smallest correlations in the model are 0.11 and 0.18 respectively for the $\sigma = 1$ case. The correlation of velocity with inflation only comes somewhat close for the case $\sigma = 1$.

as Hodrick et al. [1991], omitted the examination of its empirical properties.

The correlation of inflation with consumption growth is almost absent in the data while it is strongly negative in the model. The same thing happens in the RA model. The opposite occurs with the correlation between inflation and money growth which is negative in the data and quite strongly positive in the model. The correlation between inflation and the nominal interest rate appears to come close to the data only for low levels of risk aversion (the correlation in the data is 0.35 and the model for the $\sigma = 1$ case gives 0.18). The strong, negative correlation of inflation and the real interest rate in the data is captured quite well by the model. The correlation between inflation and the return on equity is a complete miss, the data shows quite a strong positive correlation and is negative in the model.

The correlation of the return on equity and consumption growth is low and positive in the data, 0.12. The model predicts, in general, a stronger association, the lowest being 0.68 for $\sigma = 3$. The correlation of the return on equity and money growth is quite well captured by the model, especially for high levels of risk aversion. For example, the data shows -0.58 for this correlation, and for $\sigma = 2$ the model delivers -0.59. The correlation of the return on equity with the real interest rate is strongly negative in the data and strongly positive in the model, so again, we have a miss. Finally, the data shows a lack of correlation between the return on equity and the nominal interest rate, and the model comes close only for low values of risk aversion. The data shows -0.02 for this correlation, while the model gives 0.11 for $\sigma = 1$.

5.3 Persistence

Table 6 shows autocorrelations for some variables of the model. First, the autocorrelation of velocity is quite well captured by the model in the first sub-sample and less so in the second sub-sample. The autocorrelation of the nominal interest rate is well captured, especially in the first sub-sample. The autocorrelation of the real interest rate is well captured for high relative

Table 5: Correlations

Variable	Data	$\sigma = 1$ HA	$\sigma = 1$ RA	$\sigma = 2$ HA	$\sigma = 2$ RA	$\sigma = 3$ HA	$\sigma = 3$ RA
A. 1984:I-2007-IV							
$corr(\vartheta, \gamma_y)$	-0.01	-0.02	0	-0.08	0	-0.13	0
$corr(\vartheta, \gamma_m)$	-0.14	-0.10	0	-0.03	0	-0.07	0
$corr(\vartheta, i)$	0.73	1.00	0	1.00	0	1.00	0
$corr(\vartheta, r_b)$	0.29	0.06	0	0.10	0	0.10	0
$corr(\vartheta, r_e)$	-0.09	-0.03	0	-0.05	0	-0.06	0
$corr(\vartheta, \pi)$	0.24	0.15	0	0.21	0	0.20	0
$corr(\pi, \gamma_y)$	-0.32	-0.97	-0.56	-0.90	-0.48	-0.89	-0.59
$corr(\pi, \gamma_m)$	-0.21	0.16	0.81	0.36	0.86	0.40	0.79
$corr(\pi, i)$	0.26	0.15	0.59	0.21	0.55	0.20	0.46
$corr(\pi, r_b)$	-0.73	-0.98	-0.83	-0.95	-0.84	-0.95	-0.86
$corr(\pi, r_e)$	-0.19	-0.97	-0.55	-0.91	-0.57	-0.92	-0.71
$corr(r_e, \gamma_y)$	0.21	1.00	1.00	0.99	0.99	0.98	0.96
$corr(r_e, \gamma_m)$	0.06	0.04	0.05	-0.02	-0.07	-0.07	-0.14
$corr(r_e, r_b)$	0.15	0.97	0.40	0.92	0.39	0.92	0.55
$corr(r_e, i)$	-0.04	-0.03	-0.40	-0.05	-0.45	-0.06	-0.42
B. 2008:I-2019-IV							
$corr(\vartheta, \gamma_y)$	0.30	0.11	0	0.09	0	0.33	0
$corr(\vartheta, \gamma_m)$	-0.12	-0.02	0	0.03	0	-0.31	0
$corr(\vartheta, i)$	0.59	1.00	0	1.00	0	1.00	0
$corr(\vartheta, r_b)$	-0.27	0.18	0	0.26	0	0.58	0
$corr(\vartheta, r_e)$	-0.12	0.11	0	0.21	0	0.54	0
$corr(\vartheta, \pi)$	0.34	0.18	0	0.08	0	-0.39	0
$corr(\pi, \gamma_y)$	-0.04	-0.92	-0.55	-0.69	-0.56	-0.64	-0.56
$corr(\pi, \gamma_m)$	-0.38	0.42	0.91	0.81	0.91	0.80	0.91
$corr(\pi, i)$	0.35	0.18	0.52	0.08	0.21	-0.39	-0.27
$corr(\pi, r_b)$	-0.99	-0.93	-0.82	-0.94	-0.91	-0.98	-0.93
$corr(\pi, r_e)$	0.60	-0.92	-0.54	-0.91	-0.88	-0.95	-0.87
$corr(r_e, \gamma_y)$	0.12	1.00	1.00	0.83	0.76	0.68	0.68
$corr(r_e, \gamma_m)$	-0.58	-0.17	-0.15	-0.59	-0.67	-0.64	-0.69
$corr(r_e, r_b)$	-0.63	0.96	0.50	0.95	0.95	0.97	0.94
$corr(r_e, i)$	-0.02	0.11	-0.19	0.21	0.16	0.54	0.59

Notes: The table presents correlations for different variables. The first column shows (logged) HP data statistics and the rest of the columns show theoretical moments for different values of σ. Under columns HA, it is shown the results for the heterogeneous agent model developed. The model statistics for the representative agent version of the model are presented in columns labeled RA. Panel A shows moments for the period 1984.I-2007.IV while Panel B presents moments for the period 2008.I-2019.IV.

risk aversion in the first sub-sample and overstates it in the second sample. For example, in the first sub-sample, the autocorrelation is 0.16 and is 0.13 in the model for $\sigma = 3$. For the second sub-sample, the correlation is 0.20 but in this case, the smallest model value is 0.40 for $\sigma = 1$, the RA case performs somewhat better with 0.27 for this case.

The autocorrelation of the return on equity is low, 0.10 in the first sub-sample and very low but negative in the model. In the second sub-sample, this correlation is 0.28 in the data, and the model captures it quite well for $\sigma = 1$, giving 0.30.

The autocorrelation of inflation in the first sub-sample is captured very well for low values of risk aversion in the first sub-sample (the model exactly matches this correlation of 0.02 for $\sigma = 1$) and overstates it in the second sub-sample (the data shows a value of 0.19 and the closest value in the model is 0.41 for $\sigma = 1$). The RA case in both sub-sample misses this autocorrelation by a large margin.

6 Concluding remarks

In this paper, I developed a model where precautionary motives in money demand are central and are embedded in an otherwise standard monetary model in the Cash-in-Advance tradition. The model is constructed in such a way as to incorporate relevant heterogeneity in terms of how individuals decide on their portfolio allocations. This was accomplished by assuming that individuals face idiosyncratic "liquidity shocks", that induce them to hold precautionary balances and to produce a rich array of decisions regarding their portfolio allocations, when the economy undergoes different shocks.

The model was used to contrast its time series properties against the data for the US in the

Table 6: Autocorrelations

Variable	Data	$\sigma=1$ HA	$\sigma=1$ RA	$\sigma=2$ HA	$\sigma=2$ RA	$\sigma=3$ HA	$\sigma=3$ RA
A. 1984:I-2007-IV							
$corr(\vartheta', \vartheta)$	0.89	0.88	0	0.88	0	0.84	0
$corr(i', i)$	0.90	0.88	0.82	0.88	0.78	0.84	0.70
$corr(r_b', r_b)$	0.16	0.03	0.30	0.09	0.30	0.13	0.31
$corr(r_e', r_e)$	0.10	-0.03	-0.03	-0.05	-0.04	-0.05	-0.03
$corr(\pi', \pi)$	0.02	0.02	0.37	0.09	0.38	0.10	0.36
B. 2008:I-2019-IV							
$corr(\vartheta', \vartheta)$	0.74	0.92	0	0.95	0	0.84	0
$corr(i', i)$	0.64	0.92	0.97	0.95	0.94	0.84	0.78
$corr(r_b', r_b)$	0.20	0.40	0.27	0.60	0.42	0.69	0.45
$corr(r_e', r_e)$	0.28	0.30	0.29	0.67	0.60	0.74	0.67
$corr(\pi', \pi)$	0.19	0.41	0.51	0.64	0.51	0.71	0.51

Notes: The table presents autocorrelations for different variables. The first column shows (logged) HP data statistics and the rest of the columns show theoretical moments for different values of σ. Under columns HA, it is shown the results for the heterogeneous agent model developed. The model statistics for the representative agent version of the model,is presented in columns labeled RA. Panel A shows moments for the period 1984.I-2007.IV while, Panel B presents moments for the period 2008.I-2019.IV.

periods 1984.I-2007.IV and 2008.I-2019.IV. That partition of the sample was guided by the potential effects of the Great Recession in shaping interactions in the financial variables in the macro-economy, including the effects of nontraditional monetary policy. In both samples, the model with heterogeneity delivers an improvement over almost all dimensions compared to a representative agent model where this type of heterogeneity is absent.

Overall, adding the precautionary motive to the CIA model, helped to account for volatility, correlations, and persistence of several variables traditionally taken into account when studying the performance of monetary models. The proposed model, however, cannot account for many moments examined. Importantly, the model performs markedly better in the first sub-sample than in the second. Therefore, this paper presents *prima facie* evidence that financial and macro-variables have suffered an important change that is reflected in their interactions since the Great Recession. In this regard, given the evidence found that a model of the type developed here is unable to capture many time series properties in the later period, future work could focus on incorporating nontraditional monetary policy in a similar framework. This is left for future work.

A The Description of the data

Construction of the dividend series and the data equivalent to y, is done as follows. I took data from the BEA (Bureau of Economic Analysis). Table 2.3.5 presents personal consumption expenditures and table 2.3.4 presents the correspondent price indexes. *Non-durable goods* expenditures are divided by its price index. The same is done for the series *services*, and the data equivalent to y is constructed by the sum of the two divided by the population level (CNP16OV) taken from FRED (Federal Reserve Economic Data).

To obtain the price level, nominal consumption is constructed as the sum of expenditures on non durable goods plus services taken from table 2.3.5 of BEA, and divided by the data equivalent to y constructed as explained above.

The monetary aggregate is considered M2, this series is usually used in empirical studies for its alleged stationarity, see for example Hodrick et al. [1991], Wang and Shi [2006], and Telyukova and Visschers [2013]. I took M2 from the FRED database.

Velocity is constructed by dividing nominal total consumption as the sum of expenditures on non-durable goods plus services taken from table 2.3.5 of BEA divided by M2.

The monetary aggregate used equals M2 divided by the population level (CNP16OV) taken from FRED. This series is used to construct the data equivalent to γ_m.

For nominal interest rates, I take the 3-Month Treasury Bill: Secondary Market Rate.

Finally, the series for the return on equity are constructed using (real) stock prices and dividends from Robert Shiller's database taken from http://www.econ.yale.edu/ shiller/data.htm. For more information, see Robert Shiller [1989].

B Solving the model: a Guess-and-Verify approach

The objective is to solve the problem (3.4) under (3.2). Let λ_j be the multiplier on the CIA constraint in (3.2), then the Lagrangian for the inner problem in (3.4) can be defined as:

$$\mathcal{L}(w_i, \theta_j) = \theta_j u(c_i) + \beta \mathbb{E}\mathcal{V}'(w_i') + \lambda_j(z_m w_i - c_i), \tag{B.1}$$

where $w_i' = R' w_i - R_m' c_i$ as defined in (3.2). The Karush-Khun-Tucker conditions for maximization are:

$$\theta_j u'(c_i) = \beta \mathbb{E}\mathcal{V}_w'(w_i') R_m' + \lambda_j \tag{B.2}$$

$$\lambda_j(z_m w_i - c_i) = 0, \quad c_i \leq z_m w_i, \quad \lambda_j \geq 0. \tag{B.3}$$

(B.2) is the FOC with respect to consumption and (B.3) are the complementary slackness conditions. With the Lagrangian (B.1), the Bellman equation in (3.4) is:

$$\mathcal{V}(w_i) = \max_{z_m, z_b} \sum_{j=1}^{n} \mathcal{L}(w_i, \theta_j) \chi_j. \tag{B.4}$$

At the beginning of each period, households choose z_m and z_b, the FOC's are:

$$z_m : \quad \sum_{j=1}^{n} \left[\theta_j u_c(c_i) \frac{\partial c_i}{\partial z_m} + \beta \mathbb{E}\mathcal{V}_w'(w_i') \frac{\partial w_i'}{\partial z_m} \right] \chi_j = 0 \tag{B.5a}$$

$$z_b : \quad \sum_{j=1}^{n} \left[\theta_j u_c(c_i) \frac{\partial c_i}{\partial z_b} + \beta \mathbb{E}\mathcal{V}_w'(w_i') \frac{\partial w_i'}{\partial z_b} \right] \chi_j = 0. \tag{B.5b}$$

Finally, the Envelope Condition is:

$$\mathcal{V}_w(w_i) = \beta \sum_{j=1}^{n} \mathbb{E}\mathcal{V}_w'(w_i') \chi_j R' + \sum_{j=1}^{n} \lambda_j z_m \chi_j. \tag{B.6}$$

Using the definition of R' the envelope condition can be written as:

$$\mathcal{V}_w(w_i) = \beta \sum_{j=1}^{n} \mathbb{E}\mathcal{V}'_w(w'_i)\chi_j[R'_e(1-z_b-z_m)+R'_b z_b] + \beta \sum_{j=1}^{n} \mathbb{E}\mathcal{V}'_w(w'_i)\chi_j R'_m z_m + \sum_{j=1}^{n} \lambda_j z_m \chi_j. \quad \text{(B.7)}$$

Then, multiplying equation (B.2) by $z_m\chi_j$, and summing over j, we get:[14]

$$\beta \sum_{j=1}^{n} \mathbb{E}\mathcal{V}'_w(w'_i)\chi_j R'_m z_m = \sum_{j=1}^{n} \theta_j u(c_i) z_m \chi_j - \sum_{j=1}^{n} \lambda_j z_m \chi_j. \quad \text{(B.8)}$$

In addition, by using (B.8), we can write (B.7) as:

$$\mathcal{V}_w(w_i) = \sum_{j=1}^{n} \theta_j u(c_i) z_m \chi_j + \beta \sum_{j=1}^{n} \mathbb{E}\mathcal{V}'_w(w'_i)\chi_j[R'_e(1-z_b-z_m)+R'_b z_b]. \quad \text{(B.9)}$$

This version of the Envelope Condition will be important later on, in the verification step of the Guess-and-Verify solution.

To solve the model the guess (3.6) is used along with the guess for the policy function for consumption:

$$c_{i,j} = (1-\zeta_j) z_m w_i. \quad \text{(B.10)}$$

Under these guesses, the FOC (B.2) is:

$$\theta_j[(1-\zeta_j)z_m]^{-\sigma} w_i^{-\sigma} = \beta \mathbb{E}\psi'(R'_j)^{-\sigma} R'_m w_i^{-\sigma} + \lambda_j. \quad \text{(B.11)}$$

Proposition 1. *Under the guess for the value function* (3.6) *and* (B.10)*, household's assets demands are homogenous fractions of their wealth.*

[14]Note that $\mathcal{V}'_w(w'_i)$ does depend on j as w'_i is a function of c_i, which depends on θ_j.

Proof. The idea is to show that optimal values of z_m and z_b do not depend on wealth w_i. Note that using the policy (B.10), wealth next period equals $w_i' = R_j' w_i$ where R_j' is defined as in (3.8). With these results, the FOC's in (B.5) are written as:[15]

$$\sum_{j=1}^{n} \left\{ \theta_j [(1-\zeta_j) z_m w_i]^{-\sigma} \frac{\partial \overline{(1-\zeta_j) z_m}}{\partial z_m} w_i + \beta \mathbb{E} \psi' (R_j' w_i)^{-\sigma} \left(-R_e' + R_m' \frac{\partial \overline{\zeta_j z_m}}{\partial z_m} \right) w_i \right\} \chi_j = 0 \tag{B.12}$$

$$\sum_{j=1}^{n} \left\{ \theta_j [(1-\zeta_j) z_m w_i]^{-\sigma} \frac{\partial (1-\zeta_j)}{\partial z_b} z_m w_i + \beta \mathbb{E} \psi' (R_j' w_i)^{-\sigma} \left(-R_e' + R_b' + R_m' \frac{\partial \zeta_j}{\partial z_b} z_m \right) w_i \right\} \chi_j = 0. \tag{B.13}$$

We can see in these equations that w_i can be canceled out from both. We must show that ζ_j does not depend on w_i either, but this is straightforward. First, when the CIA binds, then we know that $\zeta_j = 0$. For those cases where $\zeta_j > 0$, equation (B.2) (in which $\lambda_j = 0$), can be written from (B.11) as:

$$\theta_j [(1-\zeta_j) z_m]^{-\sigma} = \beta \mathbb{E} \psi' (R_j')^{-\sigma} R_m'. \tag{B.14}$$

This equation shows, that ζ_j does not depend on w_i. □

Proposition 2. *The stochastic process for* ψ *in* (3.6) *is given by:*

$$\psi = \sum_{j=1}^{n} \left\{ \theta_j [(1-\zeta_j) z_m]^{1-\sigma} + \beta \mathbb{E} \psi' (R_j')^{1-\sigma} \right\} \chi_j. \tag{B.15}$$

Proof. This step amount to the verification of the Guess for the value function in (3.6). For this, we use (B.4). Note that when the CIA binds, the complementary slackness conditions (B.3) imply that $\lambda_j (z_m w_i - c_i) = 0$, then (B.4) is satisfied as:

$$\mathcal{V}(w_i) = \sum_{j=1}^{n} \left\{ \theta_j \frac{[(1-\zeta_j) z_m w_i]^{1-\sigma}}{1-\sigma} + \beta \mathbb{E} \psi' \frac{(R_j' w_i)^{1-\sigma}}{1-\sigma} \right\} \chi_j. \tag{B.16}$$

[15] Here, the bar over the product of variables such as $\overline{(1-\theta_j) z_m}$, denotes the *vinculum*.

Making explicit the value function in the LHS of this equation and factorizing terms:

$$\psi\frac{w_i^{1-\sigma}}{1-\sigma} = \left\{\sum_{j=1}^{n}\left[\theta_j(1-\zeta_j)^{1-\sigma}z_m^{1-\sigma} + \beta\mathbb{E}\psi'(R_j')^{1-\sigma}\right]\chi_j\right\}\frac{w_i^{1-\sigma}}{1-\sigma}. \tag{B.17}$$

Note that because of previous results, none of the terms in braces in (B.17) depend on w_i. Therefore, the guess is validated equating ψ to the term in braces, which gives equation (B.15), the same as equation (3.18) in the text. □

Finally, we go to the verification step for the policy function for consumption. This step is necessary because two guesses have been employed, one for the value function and the other for the policy function for consumption. We need to check, if under all these guesses, the Envelope Condition (B.9) is satisfied.

Replacing in (B.9) the guesses and using equation (3.8):

$$\mathcal{V}_w(w_i) = \sum_{j=1}^{n}\theta_j[(1-\zeta_j)z_m w_i]^{-\sigma}z_m\chi_j + \beta\sum_{j=1}^{n}\mathbb{E}\psi'(R_j'w_i)^{-\sigma}(R_j' - R_m'\zeta_j z_m), \tag{B.18}$$

which equals:

$$\psi w_i^{-\sigma} = \left\{\sum_{j=1}^{n}\theta_j[(1-\zeta_j)z_m]^{-\sigma}z_m\chi_j + \beta\sum_{j=1}^{n}\mathbb{E}\psi'(R_j')^{1-\sigma}\chi_j - \beta\sum_{j=1}^{n}\mathbb{E}\psi'(R_j')^{-\sigma}R_m'\zeta_j z_m\chi_j\right\}w_i^{-\sigma}. \tag{B.19}$$

Eliminating $w_i^{-\sigma}$ from both sides, we have:

$$\psi = \sum_{j=1}^{n}\theta_j[(1-\zeta_j)z_m]^{-\sigma}z_m\chi_j + \beta\sum_{j=1}^{n}\mathbb{E}\psi'(R_j')^{1-\sigma}\chi_j - \sum_{j=1}^{n}\theta_j[(1-\zeta_j)z_m]^{-\sigma}\zeta_j z_m\chi_j. \tag{B.20}$$

Where in the last term I used (B.14) and the fact that $\zeta_j = 0$ when CIA binds. We can group the first and the third term in the last equation to get:

$$\psi = \sum_{j=1}^{n} \left\{ \theta_j [(1-\zeta_j) z_m]^{1-\sigma} + \beta \mathbb{E} \psi' (R'_j)^{1-\sigma} \right\} \chi_j. \tag{B.21}$$

which replicates (B.15) and therefore the guesses employed are internally consistent.

B.1 The system

Here I collect the relevant equations of the model. Some need to be normalized as the endowment y perpetually grows even in steady state, these normalized variables will be denoted with a "hat". Returns from equations (3.3) are:

$$R'_e = \left[\hat{q}'_e (1+\gamma'_y) + \frac{1}{1+\pi'} \right] \frac{1}{\hat{q}_e}, \quad R'_b = \frac{1}{(1+\pi')Q_b}, \quad R'_m = \frac{1}{1+\pi'}. \tag{B.22a}$$

Portfolio returns after consumption, from (3.8), are:

$$R_j = R'_e (1 - z_b - z_m) + R'_b z_b + R'_m \zeta_j z_m, \quad j = 1, 2, ..., n. \tag{B.22b}$$

Changes in this return when z_m, and z_b change, from (3.11) and (3.14):

$$\frac{\partial R'_j}{\partial z_m} = R'_e - R'_m \left(\zeta_j + \frac{\partial \zeta_j}{\partial z_m} z_m \right), \quad j = 1, ..., n-1. \quad \frac{\partial R'_j}{\partial z_m} = R'_e, \quad j = n. \tag{B.22c}$$

$$\frac{\partial R'_j}{\partial z_b} = R'_e - R'_b - R'_m \frac{\partial \zeta_j}{\partial z_b} z_m, \quad j = 1, ..., n-1. \quad \frac{\partial R'_j}{\partial z_b} = R'_e - R'_b, \quad j = n. \tag{B.22d}$$

The change in saving rates when z_m and z_b change, from (3.12) and (3.15):

$$\frac{\partial \zeta_j}{\partial z_m} = \frac{\theta_j z_m^{-(\sigma+1)}(1-\zeta_j)^{-\sigma} + \beta \mathbb{E}\psi'(R_j')^{-(\sigma+1)} R_m' \left(R_e' - \zeta_j R_m'\right)}{\theta_j (1-\zeta_j)^{-(\sigma+1)} z_m^{-\sigma} + \beta \mathbb{E}\psi'(R_j')^{-(\sigma+1)}(R_m')^2 z_m} \tag{B.22e}$$

$$\frac{\partial \zeta_j}{\partial z_b} = \frac{\beta \mathbb{E}\psi'(R_j')^{-(\sigma+1)} R_m'(R_e' - R_b')}{\theta_j (1-\zeta_j)^{-(\sigma+1)} z_m^{-\sigma} + \beta \mathbb{E}\psi'(R_j')^{-(\sigma+1)}(R_m')^2 z_m}. \tag{B.22f}$$

Equations determining optimal saving rates, from (3.9):

$$\theta_j[(1-\zeta_j)z_m]^{-\sigma} = \beta \mathbb{E}\psi'(R_j')^{-\sigma} R_m', \quad j = 1, ..., n-1. \tag{B.22g}$$

Optimality of z_m, from (3.16):

$$\sum_{j=1}^{n-1} \theta_j [(1-\zeta_j)z_m]^{-\sigma}\left(1 - \zeta_j - \frac{\partial \zeta_j}{\partial z_m} z_m\right)\chi_j + \theta_n z_m^{-\sigma}\chi_n - \beta \mathbb{E}\psi' \sum_{i=1}^{n} (R_j')^{-\sigma} \frac{\partial R_j'}{\partial z_m}\chi_j = 0. \tag{B.22h}$$

Optimality of z_b, from (3.17):

$$\sum_{j=1}^{n-1} \theta_j [(1-\zeta_j)z_m]^{-\sigma}\left(-\frac{\partial \zeta_j}{\partial z_b} z_m\right)\chi_j - \beta \mathbb{E}\psi' \sum_{j=1}^{n} (R_j')^{-\sigma} \frac{\partial R_j'}{\partial z_b}\chi_j = 0. \tag{B.22i}$$

The recursion for ψ, from (3.18):

$$\psi = \sum_{j=1}^{n} \theta_j [(1-\zeta_j)z_m]^{1-\sigma}\chi_j + \beta \mathbb{E}\psi' \sum_{i=1}^{n} (R_j')^{1-\sigma}\chi_j. \tag{B.22j}$$

Government budget constraint, from (3.21):

$$\left(\frac{z_b}{Q_b} + z_m\right)\hat{w} = (1+\pi')(z_m' + z_b')\hat{w}'(1+\gamma_y'). \tag{B.22k}$$

Money creation, from (3.22):

$$z_m'\hat{w}'(1+\gamma_y') = \frac{1+\mu'}{1+\pi'} z_m \hat{w}. \tag{B.22l}$$

Goods market clearing, using the last equation in (2.10) and the first equation in (3.19):

$$1 = \sum_{j=1}^{n} (1 - \zeta_j)\chi_j z_m \hat{w}. \tag{B.22m}$$

Equity market clearing, from the equity market in equations (2.10) and the last equation in (3.20):

$$\hat{q}_e = (1 - z_b - z_m)\hat{w}. \tag{B.22n}$$

The system is closed with the stochastic processes for money and the dividend growths in (2.9). Note that the variable x can be written as: $x = z_b\hat{w}/Q_b$. Mechanically speaking, under some parametrization, equations (B.22g) to (B.22n) for a system of 11 equations in the unknowns: $\hat{q}_e, \pi, Q_b, z_m, z_b, \psi, \hat{w}, \zeta_1, ..., \zeta_4$.

B.2 The steady state

In this section, I present the model's analysis for the steady state where all aggregate shocks are muted, but idiosyncratic shocks continuously buffet agents. This will be helpful to understand the workings of the model and also helpful for the model's calibration.

In the (aggregate) non-stochastic steady state the rate of returns on both bonds and equity must be the same $\bar{R}_e = \bar{R}_b$. This is confirmed by examining equations (B.22i) and (B.22f), such a rate will be denoted $\bar{R}$. The first two equations in (B.22a) at steady state, therefore, yield to equation (4.2) in the text.

After consumption returns are from (B.22b):

$$\bar{R}_j = \bar{R}(1 - \bar{z}_m) + \bar{R}_m\bar{\zeta}_j\bar{z}_m, \quad j = 1, 2, ..., n. \tag{B.23}$$

Changes in the portfolio return with changes in $\bar{z}_m$, from (B.22c):

$$\frac{\partial \bar{R}_j}{\partial \bar{z}_m} = \bar{R} - \bar{R}_m \left(\bar{\zeta}_j + \frac{\partial \bar{\zeta}_j}{\partial \bar{\zeta}_m} \bar{z}_m \right), \quad j = 1, ..., n-1 \quad \frac{\partial \bar{R}_j}{\partial \bar{z}_m} = \bar{R}, \quad j = n. \tag{B.24}$$

The change in saving rates when $\bar{z}_m$ and $\bar{z}_b$ change, from (B.22e) and (B.22f):

$$\frac{\partial \bar{\zeta}_j}{\partial \bar{z}_m} = \frac{\theta_j \bar{z}_m^{-(\sigma+1)} (1 - \bar{\zeta}_j)^{-\sigma} + \beta \bar{\psi} (\bar{R}_j)^{-(\sigma+1)} \bar{R}_m \left(\bar{R} - \bar{\zeta}_j \bar{R}_m \right)}{\theta_j (1 - \bar{\zeta}_j)^{-(\sigma+1)} \bar{z}_m^{-\sigma} + \bar{\psi} (\bar{R}_j)^{-(\sigma+1)} (\bar{R}_m)^2 \bar{z}_m}, \quad \frac{\partial \bar{\zeta}_j}{\partial \bar{z}_b} = 0. \tag{B.25}$$

Equations determining optimal saving rates are given from (B.22g), in steady state, in equations (4.3a) in the text.

Optimality of z_m and the value of ψ from (B.22h) and (B.22i) are given in equations (4.3b) and (4.3c) in the text.

Government budget constraint from (B.22k) in steady state is given by equation (4.3d) in the text. Money growth equation, from (B.22l) in steady state is given by:

$$1 + \bar{\gamma}_y = \frac{1 + \bar{\gamma}_m}{1 + \bar{\pi}}. \tag{B.26}$$

Finally, the goods market clearing equation (B.22m) and equity market clearing (B.22n) can be used to eliminate $\bar{\bar{w}}$ and get the equation (4.3e) in the text.

B.2.1 An intuitive explanation

Here I use the model in steady state to give an intuitive explanation about the workings of the model using equations (4.3a) and (B.23). I illustrate households' decisions in the second subperiod, where they choose $\bar{\zeta}_j$. Equation (B.23) can be written as:

$$\bar{R}_m(1-\bar{\zeta}_j)\bar{z}_m + \bar{R}_j = \bar{R}(1-\bar{z}_m) + \bar{R}_m\bar{z}_m. \tag{B.27}$$

Note that the term $(1-\bar{\zeta}_j)\bar{z}_m$ is consumption per unit of wealth, so this equation says how "income" $\bar{R}(1-\bar{z}_m)+\bar{R}_m\bar{z}_m$ is allocated among two "goods": consumption per unit of wealth and the return on the portfolio after consumption $\bar{R}_j$. The price of consumption is $\bar{R}_m$ because a higher inflation rate will lower the price of current consumption as future consumption needs to be paid with cash. Equation (4.3a) then gives then a familiar relationship in price theory (here allowing for a binding CIA constraint):

$$\frac{\theta_j}{\beta\bar{\psi}}\left[\frac{(1-\bar{\zeta}_j)\bar{z}_m}{\bar{R}_j}\right]^{-\sigma} \geq \bar{R}_m, \tag{B.28}$$

which states that the Marginal Rate of Substitution (MRS) is greater or equal to the price of consumption. When CIA binds then this equation is satisfied with strict inequality. Figure 2 shows the budget constraint (B.27). The curve $\mathcal{C}_1$ shows an "indifference curve" yielding an interior solution in which (B.28) is satisfied with equality. This will happen when the liquidity shock is relatively low, or when β and $\bar{\psi}$ are relatively high. The intuition is straightforward. A low liquidity shock induces individuals to hold cash for the future, and they substitute consumption in favor of a higher return $\bar{R}_j$. A high liquidity shock induces households to bind their CIA constraint in which $\bar{\zeta}_j = 0$, this will correspond to an "indifference curve" $\mathcal{C}_2$ in the figure. Higher patience or more valuation for next period wealth (higher β or $\bar{\psi}$) induce individuals to substitute consumption towards more cash for next period.

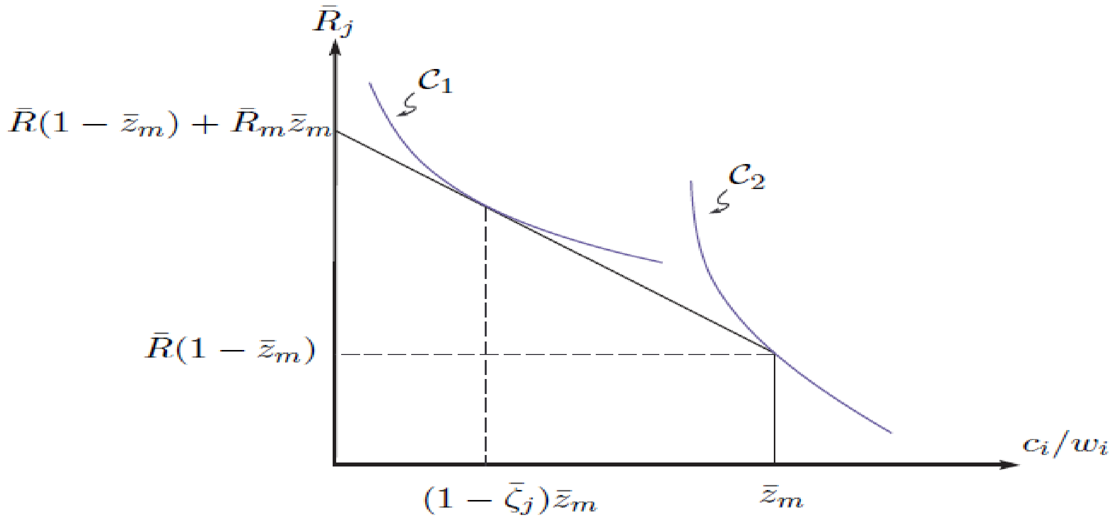

Figure 2: When MRS equals $\bar{R}_m$, households choose an interior optimum in which cash is stored for the next period. When MRS is higher than $\bar{R}_m$, households will choose to deplete all cash purchasing consumption, in that case, the CIA binds.

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